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Adjoint symmetries and the generation of first integrals in non-holonomic mechanics

W. Sarlet[a,*], D.J. Saunders[b,1], F. Cantrijn[a]

[a] *Department of Mathematical Physics and Astronomy, Ghent University, Krijgslaan 281, B-9000 Ghent, Belgium*
[b] *The Open University, Walton Hall, Milton Keynes MK7 6BH, UK*

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Abstract

We discuss a general mechanism by which first integrals of mechanical systems, in particular systems that satisfy non-holonomic constraints, can be obtained from a systematic search for adjoint symmetries. Such an approach has already been used in our earlier work and is re-advocated here in the context of a recent analysis by Giachetta, in which first integrals are generated by vector fields which are not symmetries. Further advantages of our approach are: the fact that an essential projection operator associated to the constraints need not be related to some given fibre metric on the full evolution space, and the specific selection of a connection, which is naturally associated to this projection and the second-order dynamics on the constraint submanifold. The computational aspects of the method are illustrated by some simple examples.

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* Corresponding author. Tel.: +32 9 2644797; fax: +32 9 2644989.
[1] Address for correspondence: 30 Little Horwood Road, Great Horwood, Milton Keynes MK17 0QE, UK.
E-mail address: willy.sarlet@ugent.be (W. Sarlet).

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1. Introduction

Over the past 30 years, a variety of differential geometric models has been developed for the description and study of non-holonomic systems; for a detailed bibliography, we refer to two recent books on the subject [2,6]. Naturally, one of the issues which has attracted attention in these studies is that of symmetry and reduction (see, e.g. [1,3,5,8]). However, if a related unconstrained system has an easily identifiable symmetry group, such a symmetry could well be destroyed by the constraint equations. Moreover, symmetries of a non-holonomic system need not give rise to a reduction of the system, nor to an induced conservation law. The situation is somewhat reminiscent of the case of non-conservative Lagrangian systems, where a generalization of Noether's theorem exists, which establishes a one-to-one correspondence between first integrals and a class of vector fields which are not symmetries (see [4]) and were later called *pseudo-symmetries* in [12].

The immediate source of inspiration for the present paper is a contribution by Giachetta [7], who indeed shows that the idea of pseudo-symmetries can be translated to the situation of non-holonomic systems, leading again to a one-to-one correspondence between first integrals and certain vector fields which are generally not symmetries. The point we wish to make, however, is that the idea of generating first integrals through pseudo-symmetries for non-conservative systems was abandoned in the work cited above [12], in favour of a dual concept of *adjoint symmetries*. Roughly speaking, the advantage of an algorithm which generates first integrals through the construction of adjoint symmetries is that it is universal: it remains unaltered when passing from classical conservative mechanical systems to non-conservative (or indeed non-holonomic) systems; the computational complexity, moreover, is the same as the one for constructing Noether symmetries or pseudo-symmetries. Adjoint symmetries are essentially invariant 1-forms, and they generate a first integral whenever they are exact, in an appropriate sense. An additional benefit is that one can occasionally obtain a surprise result in searching for such adjoint symmetries, namely that one can obtain a Lagrangian for a given dynamical system, which was not previously known to have one. This covers the well-known situation that a so-called non-Noether symmetry of a Lagrangian system gives rise to an alternative Lagrangian, possibly trivial or degenerate, however.

Our study of non-holonomic systems in [13,14] already established a theorem which is the analogue of the one on adjoint symmetries known from previous work. Our claim now is that such a theorem must contain all the information which Giachetta obtained when he extended the theory of pseudo-symmetries. Establishing this statement is the main objective of the present paper. Both [7,14] rely on the existence of a projection operator from vectors vertical on the full evolution space, to vertical vectors tangent to the constraint submanifold. But the projection mechanism in Giachetta's paper is entirely different from ours, and we shall demonstrate here that the ideas underlying our results on adjoint symmetries are independent of the method of projection which is being used. Another point of difference is the general formalism which we use: it is based on an adapted calculus along the projection π_C from the constraint manifold C onto the configuration space; the advantage of such an approach is that it provides the most economic way of modelling the underlying analytical calculations in a coordinate-free way.

The structure of this paper is as follows. First, we briefly review the classical concept of an adjoint symmetry and take the opportunity to sketch in some more detail what the general merits are of our specific calculus. The geometrical foundations of our approach in the present context of non-holonomic mechanics are explained in Section 3, and in Section 4, where we introduce a particular connection which will simplify the calculation of adjoint symmetries. The theory of symmetries and adjoint symmetries is developed in Section 5. The main result about the relationship between a subclass of adjoint symmetries and first integrals follows in Section 6. In the final section, we illustrate the theory with some elementary examples.

2. Adjoint symmetries versus pseudo-symmetries

In this section, we summarise the relationship between pseudo-symmetries and adjoint symmetries, as established in [12] in the context of an autonomous Lagrangian system without constraints; a similar description can be given when the Lagrangian has explicit time dependence (see [15]).

Suppose L is a given regular Lagrangian on TM and the corresponding second-order field Γ is determined by

$$i_\Gamma \mathrm{d}(S^*(\mathrm{d}L)) = -\mathrm{d}E_L, \tag{1}$$

where S denotes the canonical vertical endomorphism on TM, S^* is the notation for its dual action on 1-forms, and E_L is the energy function associated to L. Let Y be a vector field on TM and put $i_Y \mathrm{d}(S^*(\mathrm{d}L)) = \alpha$. Then, $i_{[\Gamma,Y]}\mathrm{d}S^*(\mathrm{d}L) = \mathcal{L}_\Gamma \alpha$, so that Y is a symmetry of Γ if and only if, trivially, α is an invariant 1-form. Noether symmetries are a subclass of the symmetries of Γ characterized by the property that $\mathcal{L}_Y(S^*(\mathrm{d}L)) = \mathrm{d}f$ for some function f and it is well known that all (time-independent) first integrals of a Lagrangian system can be associated to such symmetries. For practical applications, one can search for Noether symmetries Y in a certain algorithmic way, but the point to be observed here is that one can equally well conduct a search for invariant 1-forms α directly.

It was established in [4] that for non-conservative systems, first integrals can still be put into direct correspondence with vector fields Y, but in general these are no longer symmetries. For this reason they were called pseudo-symmetries in [12], and can be defined as follows. A system with Lagrangian L and extra non-conservative forces with generalized components Q_i, has the property that there is a 1-form ϕ satisfying the relation

$$\mathcal{L}_\Gamma(S^*(\phi)) = \phi; \tag{2}$$

in fact, $\phi = \mathrm{d}L + Q_i(q,v)\mathrm{d}q^i$ is such a form. A vector field $Y \in \mathfrak{X}(TM)$ is then said to be a pseudo-symmetry (with respect to ϕ) if

$$i_{[Y,\Gamma]}\mathrm{d}S^*(\phi) = i_Y \mathrm{d}\phi. \tag{3}$$

It is called a pseudo-symmetry of Noether type if both $\mathcal{L}_Y(S^*(\phi)) = \mathrm{d}f$ for some f, and also $i_Y(\phi - \mathrm{d}i_\Delta \phi) = 0$ (where Δ is the dilation field on TM); indeed (3) follows from these conditions. But as observed in [12], if we now put $\alpha = i_Y \mathrm{d}S^*(\phi)$, the requirement (3) simply expresses the fact that $\mathcal{L}_\Gamma \alpha = 0$, as before. Hence, from the dual point of view, nothing has changed; we are just looking for invariant 1-forms all the time.

To justify the terminology *adjoint symmetry*, the point is that the essential second-order pdes which have to be solved for the determination of adjoint symmetries do indeed constitute the adjoint equations (in the sense in which this is understood in the theory of partial differential equations) of the equations for symmetries of Γ. Observe in addition that both these sets of pdes are, roughly speaking, equations for only half of the components of the corresponding 1-form or vector field on *TM*, the other components then being determined automatically. The coordinate-free interpretation of this feature is that we are thus looking at conditions for the determination of 1-forms or vector fields along the tangent bundle projection, rather than on *TM* itself. For this reason, in more recent intrinsic descriptions of adjoint symmetries, we have put the theory directly in the context of derivations of (scalar- and vector-valued) forms along this projection (or the appropriate generalization needed for the situation at hand). This approach first requires the availability of a connection, which gives rise to horizontal lifting procedures complementing the naturally available vertical lift, and to corresponding horizontal and vertical exterior derivatives. All theoretical results can be derived in such a reduced set-up; then, if necessary, an appropriate lift of the adjoint symmetry, regarded as 1-form along a map, will give rise to an invariant 1-form on the full space.

We repeat that the main advantage of the adopted formalism is that it gives intrinsic equations which model directly the pdes which will have to be solved in applications; the corresponding lifted objects, although geometrically equally important, provide in a sense a double set of equations, half of which relate to redundant components. For a full account of the theory of derivations along the tangent bundle projection, we refer to [10,11]. Elements of such a calculus, suitably adapted to the case where non-holonomic constraints are involved, were already used in [13,14]. We begin, in the next section, by setting up the structures which are needed to approach the specific situation studied by Giachetta from the same point of view.

3. A geometric structure for non-holonomic mechanics

From now on, we consider the time-dependent case, and take an $(n+1)$-dimensional configuration manifold E and a fibration $\tau : E \to \mathbf{R}$. We shall consider dynamical systems defined, not on the whole of the first jet space $J^1\tau$ of τ, but rather on some closed submanifold $C \subset J^1\tau$, where $\dim J^1\tau = 2n+1$, $\dim C = n+m+1$; so C is the *constraint manifold* of the system. If we let $\pi : J^1\tau \to E$ be the induced jet projection then we may put $\pi_C : C \to E$ for the restriction of π; we shall consider only those systems where π_C is a sub-bundle (not necessarily affine). In general, C will not be the jet space of a submanifold of E, and then the constraints will be *non-holonomic*.

Let $(t, q^i, \dot{q}^i)$ be coordinates on $J^1\tau$ and (t, q^i, z^a) coordinates on C (where $a = 1, 2, \dots, m$), so that z^a are coordinates on the fibres of π_C. C can locally be described as the level set $\phi^\mu = 0$ of some functions ϕ^μ (where $\mu = 1, 2, \dots, n-m$) satisfying the non-degeneracy condition that the rank of the matrix $\partial\phi^\mu/\partial\dot{q}^i$ should be maximal at points of C. Alternatively, we shall let $\psi^i(t, q, z)$ be the coordinate representation of the inclusion mapping $\iota : C \to J^1\tau$, so that $\psi^i = \dot{q}^i \circ \iota$. We then have the identity

$$\frac{\partial \psi^i}{\partial z^a}\frac{\partial \phi^\mu}{\partial \dot{q}^i} = 0, \tag{4}$$

valid at points of C. We shall write $\hat{\theta}^i = \iota^*\theta^i$ for the pull-backs of the contact forms on $J^1\tau$, so that $\hat{\theta}^i = \mathrm{d}q^i - \psi^i \mathrm{d}t$.

The construction given by Giachetta in [7] started from the assumption that we have a fibre metric g, defined on vertical vectors on $J^1\tau$, which is in fact an assumption used by many authors, the fibre metric usually coming from the Lagrangian of a given unconstrained system. If $\hat{g}$ is the restriction of g to vertical vectors tangent to C, Giachetta defines a projection P by

$$\hat{g}(P(\xi), \eta) = g(\xi, \eta)$$

for all ξ, η which are vertical vectors at the same point of C, with η tangent to C. In coordinates, such a projection is of the form

$$P\left(\frac{\partial}{\partial \dot{q}^i}\right) = P_i^a \frac{\partial}{\partial z^a}, \quad \text{with } P_i^a = g^{ab}\frac{\partial \psi^j}{\partial z^b} g_{ij}. \tag{5}$$

Here, g_{ij}, g_{ab} are the components of g and $\hat{g}$, respectively, and $g^{ac}g_{cb} = \delta^a_b$. The reason why vertical vectors tangent to C are important is that they represent the vertical lifts of admissible virtual displacements in the sense of the Chetaev-d'Alembert principle. The metric g (and the derived projection P) further play a role in Giachetta's construction of a reduced or constrained dynamics, which is a certain second-order equation field on C, and in the definition of an associated connection. But there are other ways in which a projector such as P may occur and a representation of the dynamics on C may be obtained. In our previous work [13,14], for example, we have used a connection on an auxiliary bundle $E \to M$ in order to define the constraint manifold, and this construction also produces a projection P. In [9], a more general almost product structure on the jet manifold is used, and its restriction to vertical vectors is a projection of the same kind.

Our analysis in this paper, therefore, will start from the two essential geometrical objects which are eventually present in all models for non-holonomic systems (adopting the Chetaev-d'Alembert point of view), without making a distinction about the way they arise: a second-order differential equation field (SODE) Γ on C and a certain projector P onto vertical vectors tangent to C. Further specific features of our approach are that we shall construct a natural connection, associated to Γ and P which differs from the one used, for example, in [7], and that for reasons explained in the preceding section, we shall do most of our intrinsic calculations with vector fields and forms *along the projection* $\pi_C : C \to E$; the relationship of these to corresponding objects on the manifold C will be an important part of our story.

So, $\Gamma \in \mathfrak{X}(C)$ is required to have the properties $\langle \Gamma, \mathrm{d}t\rangle = 1$, $\langle \Gamma, \hat{\theta}^i\rangle = 0$. In coordinates,

$$\Gamma = \frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i} + f^a \frac{\partial}{\partial z^a} \tag{6}$$

for some functions f^a defined on C.

Let Γ_i^a, Γ_0^a be the connection coefficients of an arbitrary non-linear connection on the bundle π_C, so that

$$\left(\frac{\partial}{\partial q^i}\right)^H = H_i = \frac{\partial}{\partial q^i} - \Gamma_i^a \frac{\partial}{\partial z^a}, \qquad \left(\frac{\partial}{\partial t}\right)^H = H_0 = \frac{\partial}{\partial t} - \Gamma_0^a \frac{\partial}{\partial z^a}, \tag{7}$$

and a basis for $\mathfrak{X}(C)$ is given by $\{H_0, H_i, V_a\}$, where $V_a = \partial/\partial z^a$. By linearity over functions on C, the horizontal lift operation extends to a map from $\mathfrak{X}(\pi_C)$, the space of vector fields along π_C, to $\mathfrak{X}(C)$. It applies in particular to

$$\mathbf{T}_C = \frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i}, \tag{8}$$

which is the restriction to C of the canonical total-time derivative operator $\mathbf{T}$. Most of our subsequent analysis can be carried out for any choice of a connection which is *compatible with the* Sode Γ in the sense of the following definition.

Definition 1. A connection on π_C is said to be compatible with a given Sode Γ on C, if $\mathbf{T}_C^H = \Gamma$.

This compatibility simply means that

$$\Gamma_0^a = -(f^a + \psi^i \Gamma_i^a). \tag{9}$$

Now every vector field Z along π_C has a unique decomposition $Z = \langle Z, \mathrm{d}t\rangle \mathbf{T}_C + \overline{Z}$, where $\overline{Z}$ has no $\partial/\partial t$ component, and in the same way every vector field X on C has a unique decomposition $X = \langle X, \mathrm{d}t\rangle \Gamma + \overline{Z}^H + V$, where V is a vertical vector field; we may therefore write

$$\mathfrak{X}(\pi_C) = \langle \mathbf{T}_C \rangle \oplus \overline{\mathfrak{X}}(\pi_C),$$
$$\mathfrak{X}(C) = \langle \Gamma \rangle \oplus \overline{\mathfrak{X}}(C) \oplus \mathcal{V}(C),$$

and the compatibility further means that the connection is completely determined by a horizontal lift from $\overline{\mathfrak{X}}(\pi_C)$ to $\overline{\mathfrak{X}}(C)$. In [13,14], we have described a particular connection satisfying this property, and details of this for the present setting will be given again in Section 4.

Translated to the framework of vector fields along π_C, for having the kind of projection P described above, we need to identify a submodule of $\overline{\mathfrak{X}}(\pi_C)$ as follows.

Definition 2. The space of virtual displacements is the submodule $\overline{\mathfrak{X}}_C \subset \overline{\mathfrak{X}}(\pi_C)$ of vector fields along π_C, whose canonical vertical lift to $J^1\tau$ yields an element of $\mathfrak{X}(C)$.

So now, the second piece of information we assume to be given is a projection $P : \overline{\mathfrak{X}}(\pi_C) \to \overline{\mathfrak{X}}_C$, which may be considered as a tensor field along π_C. We denote the complement to $\overline{\mathfrak{X}}_C$ under the projection P by $\tilde{\mathfrak{X}}_C$, so that

$$\overline{\mathfrak{X}}(\pi_C) = \overline{\mathfrak{X}}_C \oplus \tilde{\mathfrak{X}}_C$$

and we let Q be the complementary projection $\overline{\mathfrak{X}}(\pi_C) \to \tilde{\mathfrak{X}}_C$. We shall sometimes also be explicit in our use of the corresponding inclusion maps $I : \overline{\mathfrak{X}}_C \to \overline{\mathfrak{X}}(\pi_C)$, $J : \tilde{\mathfrak{X}}_C \to \overline{\mathfrak{X}}(\pi_C)$.

Dually, denoting by $\mathfrak{X}^*(\pi_C)$ the space of 1-forms along π_C (which we can identify with the space of semi-basic 1-forms on C), we have

$$\mathfrak{X}^*(\pi_C) = \langle \mathrm{d}t \rangle \oplus \mathfrak{C}(\pi_C)$$

where $\mathfrak{C}(\pi_C)$ can be identified with the space of contact forms on C, spanned by $\hat{\theta}^i$. The dual of the decomposition of $\overline{\mathfrak{X}}(\pi_C)$ is then

$$\mathfrak{C}(\pi_C) = \overline{\mathfrak{C}}_C \oplus \tilde{\mathfrak{C}}_C$$

where $\langle \overline{\mathfrak{X}}_C, \tilde{\mathfrak{C}}_C \rangle = \langle \tilde{\mathfrak{X}}_C, \overline{\mathfrak{C}}_C \rangle = 0$. The dual maps of the inclusions I, J are projections $I^* : \mathfrak{C}(\pi_C) \to \overline{\mathfrak{C}}_C$ and $J^* : \mathfrak{C}(\pi_C) \to \tilde{\mathfrak{C}}_C$, whereas corresponding inclusion maps are $P^* : \overline{\mathfrak{C}}_C \to \mathfrak{C}(\pi_C)$ and $Q^* : \tilde{\mathfrak{C}}_C \to \mathfrak{C}(\pi_C)$. In coordinates, a local basis for the space $\overline{\mathfrak{X}}_C$ is given by the vector fields

$$Z_a = \frac{\partial \psi^i}{\partial z^a} \frac{\partial}{\partial q^i} \tag{10}$$

because $Z_a^V = V_a$. The projection P may then be determined by some functions P_j^a (not necessarily the functions (5)), for which

$$P\left(\frac{\partial}{\partial q^j}\right) = P_j^a Z_a, \quad \text{with } \frac{\partial \psi^i}{\partial z^b} P_i^a = \delta_b^a. \tag{11}$$

It follows that we can take

$$\theta^a = P_i^a \hat{\theta}^i \tag{12}$$

to be the basis of $\overline{\mathfrak{C}}_C$ dual to the basis Z_a. A local basis for the space $\tilde{\mathfrak{C}}_C$ is also easy to construct, starting with the functions ϕ^μ defining $C \subset J^1\tau$. Indeed, putting $\eta^\mu = \iota^* S^*(\mathrm{d}\phi^\mu)$, where S is the vertical endomorphism on $J^1\tau$, so that

$$\eta^\mu = \frac{\partial \phi^\mu}{\partial \dot{q}^j} \hat{\theta}^j, \tag{13}$$

we have $\langle Z_a, \eta^\mu \rangle = 0$, in view of (4). The non-degeneracy condition on the functions ϕ^μ ensures that these forms are linearly independent, and hence constitute a basis of $\tilde{\mathfrak{C}}_C$. By dimension, we know that the contact forms $\hat{\theta}^i$ are spanned by the θ^a and η^μ. Since $\langle Z_b, \hat{\theta}^i \rangle = \partial \psi^i / \partial z^b$, putting

$$\hat{\theta}^i = \frac{\partial \psi^i}{\partial z^a} \theta^a + Z_\mu^i \eta^\mu \tag{14}$$

for some functions Z_μ^i on C, it follows from (12) and (13) that

$$P_i^a Z_\mu^i = 0, \qquad \frac{\partial \phi^\nu}{\partial \dot{q}^i} Z_\mu^i = \delta_\mu^\nu, \qquad \delta_j^i = \frac{\partial \psi^i}{\partial z^a} P_j^a + Z_\mu^i \frac{\partial \phi^\mu}{\partial \dot{q}^j}. \tag{15}$$

A basis for $\tilde{\mathfrak{X}}_C$ and representation for the projection Q finally is given by

$$Z_\mu = Z^j_\mu \frac{\partial}{\partial q^j}, \qquad Q\left(\frac{\partial}{\partial q^j}\right) = \frac{\partial \phi^\mu}{\partial \dot{q}^j} Z_\mu. \tag{16}$$

The various identities which we have obtained in this discussion will frequently be used in the calculations which follow, as will the decomposition for $\partial/\partial q^i$, regarded as a vector field along π_C, namely

$$\frac{\partial}{\partial q^i} = P^a_i Z_a + \frac{\partial \phi^\mu}{\partial \dot{q}^i} Z_\mu. \tag{17}$$

We emphasize that these decompositions do not make any a priori assumptions about the relationships between the connection, the projections and the dynamical vector field, beyond the single compatibility condition $\Gamma = \mathbf{T}_C{}^H$.

4. A natural connection for non-holonomic mechanics

We shall now make a specific choice of connection associated to each pair (Γ, P). This choice uses the vertical endomorphism $\hat{S}$ on C induced by the projection P.

Definition 3. For each $X \in \mathfrak{X}(C)$, we let $\hat{S}(X)$ be the vector field on C defined by

$$\hat{S}(X) = (P(T\pi_C \circ X))^V.$$

Here, P is interpreted in an extended sense as projection $\langle \mathbf{T}_C \rangle \oplus \bar{\mathfrak{X}}(\pi_C) \to \bar{\mathfrak{X}}_C$, with $P(\mathbf{T}_C) = 0$. In coordinates, therefore,

$$\hat{S} = P^a_i V_a \otimes \hat{\theta}^i = V_a \otimes \theta^a. \tag{18}$$

We now use the tensor $\mathcal{L}_\Gamma \hat{S}$ to define our non-linear connection in a way similar to the construction in [13,14].

Theorem 1. *The tensor field P_H on C, determined by*

$$P_H = \tfrac{1}{2}(\mathrm{id} - \mathcal{L}_\Gamma \hat{S} + \Gamma \otimes \mathrm{d}t + N), \tag{19}$$

whereby

$$N = \mathrm{id} - (\mathcal{L}_\Gamma \hat{S})^2 - \Gamma \otimes \mathrm{d}t, \tag{20}$$

is the horizontal projector of a uniquely defined non-linear connection on π_C.

Proof. In coordinates, we have

$$\mathcal{L}_\Gamma \hat{S} = V_a \otimes (\Gamma(P^a_i)\hat{\theta}^i + P^a_i(\mathrm{d}\psi^i - \Gamma(\psi^i)\mathrm{d}t)) - \left(\frac{\partial \psi^i}{\partial z^a}\frac{\partial}{\partial q^i} + \frac{\partial f^b}{\partial z^a}\frac{\partial}{\partial z^b}\right) \otimes \theta^a.$$

Now, if

$$\Gamma = \mathbf{T}_C^H, \qquad X_a = Z_a^H, \qquad X_\mu = Z_\mu^H, \qquad V_a = Z_a^V$$

is a basis of vector fields on C adapted to the connection we are about to fix, with dual basis of 1-forms

$$\mathrm{d}t,\ \theta^a,\ \eta^\mu,\ \eta^a = \mathrm{d}z^a + \Gamma_i^a \left(\frac{\partial \psi^i}{\partial z^b}\theta^b + Z_\mu^i \eta^\mu \right) - f^a \mathrm{d}t,$$

one can verify that the coordinate expression for P_H can be written in the form

$$P_H = \Gamma \otimes \mathrm{d}t + X_a \otimes \theta^a + X_\mu \otimes \eta^\mu - Z_\mu^j (R_j^a - \Gamma_j^a) V_a \otimes \eta^\mu - \frac{1}{2}\frac{\partial \psi^j}{\partial z^b}(R_j^a - 2\Gamma_j^a) V_a \otimes \theta^b,$$

with

$$R_i^a = \Gamma(P_i^a) + P_j^a \frac{\partial \psi^j}{\partial q^i} - P_i^b \frac{\partial f^a}{\partial z^b}. \tag{21}$$

For this to have the required properties of a horizontal projector, its image must be the space spanned by Γ, X_a and X_μ, so that we must have

$$\frac{\partial \psi^j}{\partial z^b}\Gamma_j^a = \frac{1}{2}\frac{\partial \psi^j}{\partial z^b} R_j^a, \tag{22}$$

$$Z_\mu^j \Gamma_j^a = Z_\mu^j R_j^a. \tag{23}$$

It turns out, making use of the third of the identities (15) that these conditions imply that

$$\Gamma_i^a = R_i^a - \frac{1}{2} P_i^b \frac{\partial \psi^j}{\partial z^b} R_j^a. \tag{24}$$

Conversely, one can verify that this expression for Γ_i^a is compatible with both requirements (22) and (23), in view of the first of the identities (15) and (11). □

We remark that previous work has indicated a relationship between the decomposition of spaces of vector fields and the eigenspaces of $\mathcal{L}_\Gamma \hat{S}$. It is always the case that Γ and V_a are eigenvectors with eigenvalues 0 and 1, respectively; our choice of connection coefficients means also that X_a are eigenvectors with eigenvalue -1, and that X_μ are eigenvectors with eigenvalue 0.

5. Symmetries and adjoint symmetries

Dynamical symmetries of Γ are vector fields X on C whose Lie derivative with respect to Γ is in the span of Γ. Two such symmetries are equivalent if they differ by a multiple of Γ, and the simplest representative in each class, therefore, has no Γ-component and is strictly invariant under the flow of Γ. Any such $X \in \mathfrak{X}(C)$ can be written in the form

$$X = \overline{Z}^H + \tilde{Z}^H + \overline{Y}^V, \tag{25}$$

with $\overline{Z}, \overline{Y} \in \overline{\mathfrak{X}}_C$ and $\tilde{Z} \in \tilde{\mathfrak{X}}_C$; from now on, a vector field written with a bar (such as $\overline{Z}$) will always be an element of $\overline{\mathfrak{X}}_C$ rather than, more generally, of $\mathfrak{X}(\pi_C) = \overline{\mathfrak{X}}_C \oplus \tilde{\mathfrak{X}}_C$, and a vector field written with a tilde (such as $\tilde{Z}$) will always be an element of the complement $\tilde{\mathfrak{X}}_C$. The idea, for such a vector field X, is to compute the decomposition of the $\mathcal{L}_\Gamma$-derivative of each part. As we know from experience in previous work, this computation will introduce the essential operators which we need to develop the theory with vector fields and forms along π_C. Once we obtain the equation for symmetries in this format, the corresponding equation for adjoint symmetries will be computed by applying the usual procedure for passing to adjoint equations, and is bound to be an equation for a 1-form along π_C. We shall then verify that the equation thus obtained is conceptually the right one by proving that there is indeed an associated 1-form on C, which is invariant under the flow of Γ.

So, our programme starts by the computation of $\mathcal{L}_\Gamma Z_a^H, \mathcal{L}_\Gamma Z_\mu^H$ and $\mathcal{L}_\Gamma Z_a^V$. In fact, since Z_a and Z_μ originate as projections of $\partial/\partial q^i$, we shall first compute $\mathcal{L}_\Gamma H_i$. A direct computation leads to

$$\mathcal{L}_\Gamma H_i = -H_i(\psi^j) H_j + \hat{\Phi}_i^a V_a, \tag{26}$$

where

$$\hat{\Phi}_i^a = H_i(\Gamma_0^a) + \psi^j H_i(\Gamma_j^a) - \Gamma(\Gamma_i^a). \tag{27}$$

The appropriate way to interpret (26) is to write it as

$$\mathcal{L}_\Gamma \left(\frac{\partial^H}{\partial q^i} \right) = \left(\nabla \frac{\partial}{\partial q^i} \right)^H + \left(\hat{\Phi} \left(\frac{\partial}{\partial q^i} \right) \right)^V. \tag{28}$$

The vertical part defines a tensorial object $\hat{\Phi} : \mathfrak{X}(\pi_C) \to \overline{\mathfrak{X}}_C$, whereas the horizontal part gives rise to a derivation ∇ of degree 0 on $\mathfrak{X}(\pi_C)$, defined as follows.

Definition 4. The dynamical covariant derivative associated to the given Sode Γ is the derivation ∇ of the $C^\infty(C)$-module $\mathfrak{X}(\pi_C)$, determined by

$$\nabla F = \Gamma(F), \quad \text{for } F \in C^\infty(C),\ \nabla \frac{\partial}{\partial q^i} = -H_i(\psi^j) \frac{\partial}{\partial q^j},\ \nabla \mathbf{T}_C = 0. \tag{29}$$

Its action on the dual module $\mathfrak{X}^*(\pi_C)$ is defined by standard duality rules.

In view of the defining relations of Z_a and Z_μ, it now follows that

$$\mathcal{L}_\Gamma Z_a^H = (\nabla Z_a)^H + (\Phi Z_a)^V, \tag{30}$$

$$\mathcal{L}_\Gamma Z_\mu^H = (\nabla Z_\mu)^H + (\Lambda Z_\mu)^V, \tag{31}$$

where we have introduced the tensor fields $\Phi : \bar{\mathfrak{X}}_C \to \bar{\mathfrak{X}}_C$ and $\Lambda : \tilde{\mathfrak{X}}_C \to \bar{\mathfrak{X}}_C$, locally given by

$$\Phi = \Phi^a_b \,\theta^b \otimes Z_a, \quad \Phi^a_b = \frac{\partial \psi^i}{\partial z^b}\hat{\Phi}^a_i, \tag{32}$$

$$\Lambda = \Lambda^a_\mu \,\eta^\mu \otimes Z_a, \quad \Lambda^a_\mu = Z^i_\mu \hat{\Phi}^a_i. \tag{33}$$

The particular choice of connection which was proposed in the previous section leads to a significant simplification in the expressions for ∇Z_a and ∇Z_μ, as we shall now see.

Proposition 1. *∇ has the properties $\nabla \tilde{\mathfrak{X}}_C \subset \tilde{\mathfrak{X}}_C$ and $\nabla \bar{\mathfrak{X}}_C \subset \bar{\mathfrak{X}}_C \oplus \tilde{\mathfrak{X}}_C$, whereby $P\nabla$ is a derivation of the module $\bar{\mathfrak{X}}_C$, whereas $\Psi := Q\nabla|_{\bar{\mathfrak{X}}_C}$ is a tensorial map from $\bar{\mathfrak{X}}_C$ into $\tilde{\mathfrak{X}}_C$.*

Proof. The proof is a matter of a direct computation, of which we leave the details to the reader. One starts from (29) and the decomposition (17); using the identities (15), together with the functions R^a_i as defined by (21), the first statement follows from the property (23) (the other property (22) of our connection simplifies the $\bar{\mathfrak{X}}_C$-component of ∇Z_a, but does not cancel it out). The results of these computations, which we need for applications, in fact read

$$\nabla Z_\mu = \frac{\partial \phi^\nu}{\partial \dot{q}^i}(\Gamma(Z^i_\mu) - Z^j_\mu H_j(\psi^i))\, Z_\nu = -Z^i_\mu \left(\Gamma\left(\frac{\partial \phi^\nu}{\partial \dot{q}^i}\right) + \frac{\partial \phi^\nu}{\partial \dot{q}^j} H_i(\psi^j)\right) Z_\nu, \tag{34}$$

$$\nabla Z_a = -\left(\frac{\partial \psi^j}{\partial z^a}\Gamma^b_j + \frac{\partial f^b}{\partial z^a}\right) Z_b + \Psi^\mu_a Z_\mu, \tag{35}$$

where we have put

$$\Psi^\mu_a = \frac{\partial \phi^\mu}{\partial \dot{q}^i}\left(\Gamma\left(\frac{\partial \psi^i}{\partial z^a}\right) - \frac{\partial \psi^j}{\partial z^a} H_j(\psi^i)\right) = -\frac{\partial \psi^j}{\partial z^a}\left(\Gamma\left(\frac{\partial \phi^\mu}{\partial \dot{q}^j}\right) + H_j(\psi^i)\frac{\partial \phi^\mu}{\partial \dot{q}^i}\right). \tag{36}$$

It follows from (35), taking the derivation property of ∇ into account, that the Ψ^μ_a are actually components of a tensor field along π_C, of the form

$$\Psi = \Psi^\mu_a \,\theta^a \otimes Z_\mu, \tag{37}$$

from which the last statement follows. □

We finally come to the computation of the Lie derivative of Z^V_a for which, in view of (35), we obtain

$$\mathcal{L}_\Gamma Z^V_a = -Z^H_a + (P\nabla Z_a)^V. \tag{38}$$

More generally, for $\bar{Z} \in \bar{\mathfrak{X}}_C$ and $\tilde{Z} \in \tilde{\mathfrak{X}}_C$, it immediately follows that

$$\mathcal{L}_\Gamma \bar{Z}^H = (\nabla \bar{Z})^H + (\Phi \bar{Z})^V, \tag{39}$$

$$\mathcal{L}_\Gamma \tilde{Z}^H = (\nabla \tilde{Z})^H + (\Lambda \tilde{Z})^V, \tag{40}$$

$$\mathcal{L}_\Gamma \overline{Z}^V = -\overline{Z}^H + (P\nabla \overline{Z})^V. \tag{41}$$

Proposition 2. *Let X be a general vector field on C, of the form (25). Then*

$$\mathcal{L}_\Gamma X = 0 \Leftrightarrow \begin{cases} \nabla \overline{Z} + \nabla \tilde{Z} - \overline{Y} = 0 \\ \Phi \overline{Z} + \Lambda \tilde{Z} + P\nabla \overline{Y} = 0 \end{cases} \tag{42}$$

$$\Leftrightarrow P\nabla(P\nabla \overline{Z}) + \nabla \tilde{Z} + \Phi \overline{Z} + \Lambda \tilde{Z} + \Psi \overline{Z} = 0. \tag{43}$$

Proof. The first equivalence follows immediately from the preceding calculations, by separating the horizontal and vertical parts. Observe next that the second of the conditions (42) lives entirely on $\overline{\mathfrak{X}}_C$, whereas the first splits further into two parts by projecting under P and Q. We thus have

$$\overline{Y} = P\nabla \overline{Z}, \tag{44}$$

which simply fixes the vertical part of a symmetry vector field, and consequently

$$\nabla \tilde{Z} + \Psi \overline{Z} = 0, \tag{45}$$

$$\Phi \overline{Z} + \Lambda \tilde{Z} + P\nabla(P\nabla \overline{Z}) = 0. \tag{46}$$

In conclusion, the determining equations which have to be solved for constructing symmetries of Γ, after the redundant part $\overline{Y}$ has been eliminated, are the partial differential Eqs. (45) and (46). Since they live on complementary spaces, they can formally be added together to the single condition (43), to which we will refer as the *symmetry condition*. □

The procedure for constructing the adjoint equation now follows the standard pattern. We contract the left-hand side of (43) with a 1-form along π_C of the form $\overline{\alpha} + \tilde{\alpha}$, with $\overline{\alpha} \in \overline{\mathfrak{C}}_C, \tilde{\alpha} \in \tilde{\mathfrak{C}}_C$; tensor fields are transferred from left to right by taking adjoints; and the operator ∇ is carried over to the form side by using the duality rule

$$\langle \nabla \cdot, \cdot \rangle = \nabla \langle \cdot, \cdot \rangle - \langle \cdot, \nabla \cdot \rangle,$$

and ignoring the first term on the right (which gives rise to a boundary term in the context of the calculus of variations).

For clarity, let us list the domains and ranges of the adjoint or dual operators involved. We have the injections $P^* : \overline{\mathfrak{C}}_C \to \mathfrak{X}^*(\pi_C),\ Q^* : \tilde{\mathfrak{C}}_C \to \mathfrak{X}^*(\pi_C)$, and the projection operators $I^* : \mathfrak{X}^*(\pi_C) \to \overline{\mathfrak{C}}_C,\ J^* : \mathfrak{X}^*(\pi_C) \to \tilde{\mathfrak{C}}_C$. Furthermore, $\Phi^* : \overline{\mathfrak{C}}_C \to \overline{\mathfrak{C}}_C,\ \Lambda^* : \overline{\mathfrak{C}}_C \to \tilde{\mathfrak{C}}_C,\ \Psi^* : \tilde{\mathfrak{C}}_C \to \overline{\mathfrak{C}}_C$. Finally, the results of Proposition 1 dualize to

$$\nabla \overline{\mathfrak{C}}_C \subset \overline{\mathfrak{C}}_C, \qquad \nabla \tilde{\mathfrak{C}}_C \subset \overline{\mathfrak{C}}_C \oplus \tilde{\mathfrak{C}}_C. \tag{47}$$

Proposition 3. *The adjoint symmetry condition reads*

$$\nabla^2 \overline{\alpha} - J^* \nabla \tilde{\alpha} + \Lambda^* \overline{\alpha} + \Phi^* \overline{\alpha} + \Psi^* \tilde{\alpha} = 0, \tag{48}$$

or equivalently

$$\nabla^2\overline{\alpha} + \Phi^*\overline{\alpha} + \Psi^*\tilde{\alpha} = 0, \tag{49}$$

$$J^*\nabla\tilde{\alpha} - \Lambda^*\overline{\alpha} = 0. \tag{50}$$

Proof. Taking all properties about domain and range into account, the dualization procedure leads to the expression $\langle \overline{Z}, \nabla^2\overline{\alpha} + \Phi^*\overline{\alpha} + \Psi^*\tilde{\alpha}\rangle - \langle \tilde{Z}, J^*\nabla\tilde{\alpha} - \Lambda^*\overline{\alpha}\rangle$. Setting the terms in $\overline{Z}$ and $\tilde{Z}$ separately equal to 0 then implies that (49) and (50) must hold and these can formally be added together to produce (equivalently) the single condition (48). □

As announced at the beginning of this section, there should now be a way to associate an adjoint symmetry, i.e. a 1-form $\overline{\alpha} + \tilde{\alpha}$ along π_C, satisfying (48), with an invariant 1-form on C. The procedure for lifting 1-forms is derived from the lift of vector fields: that is to say, for the horizontal and vertical lift of a 1-form $\alpha \in \mathfrak{X}^*(\pi_C)$, the rule generally is that for all $Z \in \mathfrak{X}(\pi_C)$,

$$\langle Z^H, \alpha^H\rangle = \langle Z^V, \alpha^V\rangle = \langle Z, \alpha\rangle, \qquad \langle Z^V, \alpha^H\rangle = \langle Z^H, \alpha^V\rangle = 0.$$

This means (remembering that we make no notational distinction between forms along π_C and their interpretation as semi-basic forms on C) that we have

$$\mathrm{d}t^H = \mathrm{d}t, \quad (\theta^a)^H = \theta^a, \quad (\eta^\mu)^H = \eta^\mu, \quad (\theta^a)^V = \eta^a.$$

Proposition 4. $\overline{\alpha} + \tilde{\alpha} \in \mathfrak{X}^*(\pi_C)$ *is an adjoint symmetry if and only if the 1-form* η *on* C*, given by*

$$\eta = \overline{\alpha}^V + \tilde{\alpha}^H - (\nabla\overline{\alpha})^H, \tag{51}$$

is invariant under Γ.

Proof. Part of the proof concerns the determination of the last term in (51), so let us write $\eta = \overline{\alpha}^V + \tilde{\alpha}^H - \overline{\beta}^H$ for the time being. For $\mathcal{L}_\Gamma\eta$ to be 0, it is necessary and sufficient that its contractions with an arbitrary $\overline{Z}^V$, $\overline{Z}^H$ and $\tilde{Z}^H$ are all 0. Essentially, we are going to express this by making use of the formulas (39)–(41). We have, for example,

$$\langle \overline{Z}^V, \mathcal{L}_\Gamma\overline{\alpha}^V\rangle = \mathcal{L}_\Gamma\langle \overline{Z}, \overline{\alpha}\rangle - \langle \mathcal{L}_\Gamma\overline{Z}^V, \overline{\alpha}^V\rangle = \nabla\langle \overline{Z}, \overline{\alpha}\rangle - \langle P\nabla\overline{Z}, \overline{\alpha}\rangle = \langle \overline{Z}, \nabla\overline{\alpha}\rangle.$$

In exactly the same way, we find $\langle \overline{Z}^V, \mathcal{L}_\Gamma\tilde{\alpha}^H\rangle = -\langle \overline{Z}, \tilde{\alpha}\rangle = 0$ and $-\langle \overline{Z}^V, \mathcal{L}_\Gamma\overline{\beta}^H\rangle = -\langle \overline{Z}, \overline{\beta}\rangle$. Adding this up, we conclude that $\langle \overline{Z}^V, \mathcal{L}_\Gamma\eta\rangle = 0$ requires that $\overline{\beta} = \nabla\overline{\alpha}$.

In exactly the same way, one can verify that $\langle \tilde{Z}^H, \mathcal{L}_\Gamma\eta\rangle = 0$ requires that (50) must hold, and $\langle \overline{Z}^H, \mathcal{L}_\Gamma\eta\rangle = 0$ imposes the condition (49). □

For later use, we list the dynamical covariant derivatives of the local basis of 1-forms along π_C. First of all, it follows from (29) by duality that

$$\nabla\hat{\theta}^i = H_j(\psi^i)\hat{\theta}^j, \qquad \nabla\mathrm{d}t = 0. \tag{52}$$

Substituting for $\hat{\theta}^i$ the decomposition (14) and projecting the resulting expression under P and Q, we obtain, in view of the simplifications brought by the choice of our connection and in agreement with the properties (47),

$$\nabla\theta^b = \left(\frac{\partial\psi^j}{\partial z^a}\Gamma^b_j + \frac{\partial f^b}{\partial z^a}\right)\theta^a, \tag{53}$$

$$\nabla\eta^\nu = -\Psi^\nu_a\theta^a - \frac{\partial\phi^\nu}{\partial\dot{q}^i}(\Gamma(Z^i_\mu) - Z^j_\mu H_j(\psi^i))\,\eta^\mu. \tag{54}$$

6. A special class of adjoint symmetries

Based on our experience in [13,14], we expect that interesting classes of adjoint symmetries could be constructed from horizontal and vertical exterior derivatives of functions on C. So, we start by defining such operations on forms along π_C for the situation at hand. We shall not pursue the development of the theory of derivations of forms along π_C in any detail here; instead we limit ourselves to the bare essentials for doing calculations. For $F \in C^\infty(C)$, we define

$$\mathrm{d}^V F = V_a(F)\theta^a, \tag{55}$$

$$\mathrm{d}^H F = \Gamma(F)\mathrm{d}t + H_i(F)\hat{\theta}^i = \Gamma(F)\mathrm{d}t + X_a(F)\theta^a + X_\mu(F)\eta^\mu. \tag{56}$$

One easily verifies in coordinates that

$$\mathrm{d}F = (\mathrm{d}^V F)^V + (\mathrm{d}^H F)^H. \tag{57}$$

When an adjoint symmetry $\alpha = \bar{\alpha} + \tilde{\alpha} \in \mathfrak{X}^*(\pi_C)$ is generated by a function F, it is clear that $\bar{\alpha}$ is likely to be of the form $\mathrm{d}^V F$, whereas $\tilde{\alpha}$ should arise from $\mathrm{d}^H F$ (explicitly, $\tilde{\alpha} = J^*\mathrm{d}^H F$). So, to verify under which circumstances such an α satisfies the adjoint symmetry condition, we need information about the way the dynamical covariant derivative ∇ commutes with the exterior derivatives. The following commutator relations follow from a straightforward coordinate calculation, making use of the bracket formulas (30), (31), (38) and the covariant derivatives (53) and (54):

$$\nabla\mathrm{d}^V F - \mathrm{d}^V\nabla F = -I^*\mathrm{d}^H F, \tag{58}$$

$$\nabla(J^*\mathrm{d}^H F) - J^*\mathrm{d}^H\nabla F = \Lambda^*\mathrm{d}^V F - \Psi^* J^*\mathrm{d}^H F, \tag{59}$$

$$\nabla(I^*\mathrm{d}^H F) - I^*\mathrm{d}^H\nabla F = \Phi^*\mathrm{d}^V F + \Psi^* J^*\mathrm{d}^H F. \tag{60}$$

It is of some interest to compare these results with the very similar formulas in [14], but we will not pursue this here. It further follows from (59) that

$$J^*(\nabla J^*\mathrm{d}^H F) - J^*\mathrm{d}^H\nabla F = \Lambda^*\mathrm{d}^V F. \tag{61}$$

Theorem 2. *A 1-form along π_C of the form $\alpha = \mathrm{d}^V F + J^*\mathrm{d}^H F$ is an adjoint symmetry of Γ, if and only if the function $L = \nabla F = \Gamma(F)$ satisfies the equations*

$$J^*\mathrm{d}^H L = 0, \qquad I^*\mathrm{d}^H L = \nabla\mathrm{d}^V L. \tag{62}$$

Proof. Inserting the assumptions for $\bar{\alpha}$ and $\tilde{\alpha}$ into the condition (50) and making use of (61) immediately produces $J^*\mathrm{d}^H L = 0$. From (58) it follows that $\nabla\bar{\alpha} = \mathrm{d}^V \nabla F - I^*\mathrm{d}^H F$. Applying ∇ again and using (58) and (60), we obtain the second of the conditions (62) from (49). □

Our main interest here is in a mechanism which is capable, in principle, of generating all first integrals of the system. Obviously, it follows from the above considerations that for every first integral F of Γ, the 1-form $\alpha = \mathrm{d}^V F + J^*\mathrm{d}^H F$ will be an adjoint symmetry. A slight inconvenience in the theorem may seem that, conversely, not every adjoint symmetry of this form will produce a first integral. A practical implementation goes as follows: one first solves the determining equations for adjoint symmetries, with a certain ansatz about the polynomial dependence on the fibre coordinates z^a; having found an adjoint symmetry α, one checks whether $\bar{\alpha}$ can be written as $\mathrm{d}^V F$ for some F; if this is the case, one verifies whether $\tilde{\alpha}$ also has the appropriate form, while making use of the additional freedom of adding basic functions f to this F. Experience shows that, in most cases, the functions $F + f$ thus obtained will be first integrals of Γ. But what if they are not?

In coordinates, the relations (62) express that the function $L = \Gamma(F)$ will satisfy:

$$X_\mu(L) = 0, \tag{63}$$

$$\Gamma\left(\frac{\partial L}{\partial z^a}\right) = X_a(L) - \frac{\partial L}{\partial z^b}\left(\frac{\partial f^b}{\partial z^a} + \frac{\partial \psi^j}{\partial z^a}\Gamma^b_j\right). \tag{64}$$

These equations are entirely similar to the ones obtained in [13]. In the framework of that paper, it is perfectly understood what it means for a non-holonomic system to be of 'Lagrangian type', and the functions L originating from the adjoint symmetry theory turn out to correspond to a subclass of such Lagrangian systems, the Lagrangian being independent of the fibre coordinates of the extra fibration. It will be an interesting topic for future studies to explore what 'variationality' means in the more general present context of a given pair (Γ, P) for the constraint submanifold. The conjecture is that, again, functions $L = \Gamma(F)$ satisfying (63) and (64) will constitute a subclass of such Lagrangian systems, provided they have a non-degenerate Hessian with respect to the z^a.

7. Some examples

As a preliminary to looking at particular examples, we shall first write down the determining equations for adjoint symmetries in coordinates. Putting $\bar{\alpha} = \alpha_a\theta^a$, $\tilde{\alpha} = \alpha_\mu\eta^\mu$, and making use of the covariant derivative expressions (53) and (54), Eq. (49) becomes

$$\Gamma^2(\alpha_a) + 2\Gamma(\alpha_b)\left(\frac{\partial \psi^j}{\partial z^a}\Gamma^b_j + \frac{\partial f^b}{\partial z^a}\right) + \alpha_b\,\Gamma\left(\frac{\partial \psi^j}{\partial z^a}\Gamma^b_j + \frac{\partial f^b}{\partial z^a}\right)$$
$$+ \alpha_b\left(\frac{\partial \psi^j}{\partial z^c}\Gamma^b_j + \frac{\partial f^b}{\partial z^c}\right)\left(\frac{\partial \psi^i}{\partial z^a}\Gamma^c_i + \frac{\partial f^c}{\partial z^a}\right) + \Phi^b_a\alpha_b + \Psi^\mu_a\alpha_\mu = 0, \tag{65}$$

and (50) reads

$$\Gamma(\alpha_\mu) - \alpha_\nu \frac{\partial \phi^\nu}{\partial \dot{q}^i}(\Gamma(Z^i_\mu) - Z^j_\mu H_j(\psi^i)) - \Lambda^a_\mu \alpha_a = 0. \tag{66}$$

Needless to say, these are quite complicated equations, but the point is that they are of the same complexity as those which will have to be solved using, for example, Giachetta's [7] procedure to search for (non-symmetry) vector fields which generate first integrals. In fact our equations become quite simple in practice, once an ansatz is made about the polynomial structure of the unknown functions with respect to the fibre coordinates. But the calculations then can still be quite tedious, so that one will be led to call in assistance of one's favourite computer algebra package.

Our first example will be one we have considered in previous work [16], namely a sled which is constrained to move so that its velocity is always in the direction of its orientation. If the coordinates on the configuration manifold $E = \mathbf{R} \times (\mathbf{R}^2 \times S^1)$ are (t, x, y, φ), where x, y represent the position of the centre of mass and φ represents orientation, then the constraint may be written in the form

$$\phi^1 = \dot{y} - \dot{x} \tan\varphi = 0$$

for most values of φ. The unconstrained equations of motion, which are of no direct relevance for our purposes, however, are generated by a Lagrangian

$$L = \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2)$$

(where we have, for simplicity, set the mass and the moment of inertia equal to 1). What matters is the dynamical vector field Γ on the constraint manifold C; taking fibre coordinates $z^1 = u = \dot{x} \circ \iota$, $z^2 = v = \dot{\varphi} \circ \iota$, it is given by

$$\Gamma = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + u \tan\varphi \frac{\partial}{\partial y} + v\frac{\partial}{\partial \varphi} - uv\tan\varphi\frac{\partial}{\partial u},$$

which exhibits also what the ψ^i are in this case. To define a projection P, it suffices to choose any functions P^a_i which satisfy (11). A convenient choice (with $a = 1, 2$ and $i = 1, 2, 3$) is given by

$$(P^a_i) = \begin{pmatrix} \cos^2\varphi & \cos\varphi\sin\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The local basis $\{Z_a\}$ follows directly from (10) while, with our choice for ϕ^1, it follows from (16) that the single element $\{Z_\mu\}$ here is given by $Q(\partial/\partial y) = (\mathrm{id} - P)(\partial/\partial y)$. The result is that

$$\left\{\frac{\partial}{\partial x} + \tan\varphi\frac{\partial}{\partial y}, \quad \frac{\partial}{\partial \varphi}\right\} \in \bar{\mathfrak{X}}_C \quad \text{and} \quad \left\{\cos^2\varphi\frac{\partial}{\partial y} - \sin\varphi\,\cos\varphi\frac{\partial}{\partial x}\right\} \in \tilde{\mathfrak{X}}_C,$$

and it is a safeguard to verify that the identities in (15) are satisfied. The final basic ingredient of our approach, namely the connection as defined by (19) and leading to the connection coefficients (24), here gives

$$(\Gamma_i^a) = \begin{pmatrix} -v\sin\varphi\,\cos\varphi & v\cos^2\varphi & u\tan\varphi \\ 0 & 0 & 0 \end{pmatrix}.$$

Concerning the tensorial quantities entering the adjoint symmetry Eqs. (65) and (66), one can verify that $\Lambda_\mu^a = 0$, while

$$(\Phi_b^a) = \begin{pmatrix} v^2 & 0 \\ -uv & 0 \end{pmatrix}, \qquad (\Psi_a^\mu) = \frac{1}{\cos^2\varphi}\begin{pmatrix} v \\ -u \end{pmatrix}.$$

Denoting the single component α_μ here, for convenience, by β, the adjoint equations now become

$$\begin{aligned} &\Gamma^2(\alpha_1) - 2v\tan\varphi\,\Gamma(\alpha_1) + \beta v\sec^2\varphi = 0, \\ &\Gamma^2(\alpha_2) - uv\alpha_1 - \beta u\sec^2\varphi = 0, \\ &\Gamma(\beta) + \beta v\tan\varphi = 0. \end{aligned}$$

An obvious particular solution is the zero solution $\alpha_1 = \alpha_2 = \beta = 0$. It produces an adjoint symmetry of the form $\mathrm{d}^V F + J^*\mathrm{d}^H F$, with $F = x + y\tan\varphi$. Clearly, such F cannot be a first integral. Instead, we easily see that $L = \Gamma(F) = \sec^2\varphi(u + vy)$ satisfies the conditions (63) and (64). Whatever the meaning of such 'surprise-Lagrangians' will turn out to be, however, it is clear that we do not really have a good example here, because this L is degenerate.

Note that the general solution of the third equation is of the form $\beta = G\cos\varphi$, where G is any first integral. This could be used to generate further adjoint symmetries, once we start obtaining first integrals. The natural assumption to start looking for particular solutions of the adjoint equations which are polynomial in the fibre coordinates is to let the α_a be functions of the base variables only and take β to be linear in u, v. One readily observes that the α_a then can be at most linear in t. Looking first for time-independent solutions, a systematic search, for which we made use of Maple, leads to six independent particular solutions:

(i) $\alpha_1 = 0, \quad \alpha_2 = 1, \quad \beta = 0;$
(ii) $\alpha_1 = 1, \quad \alpha_2 = y, \quad \beta = 0;$
(iii) $\alpha_1 = \tan\varphi, \quad \alpha_2 = -x, \quad \beta = 0;$
(iv) $\alpha_1 = \sec\varphi, \quad \alpha_2 = 0, \quad \beta = -v\cos\varphi;$
(v) $\alpha_1 = 2(y - x\tan\varphi), \quad \alpha_2 = x^2 + y^2, \quad \beta = 2u;$
(vi) $\alpha_1 = 0, \quad \alpha_2 = \varphi, \quad \beta = 0.$

The first five of these give rise to first integrals, which read, respectively:

$$F_1 = v, \qquad F_2 = u + yv, \qquad F_3 = u\tan\varphi - xv, \qquad F_4 = u\sec\varphi,$$
$$F_5 = 2u(y - x\tan\varphi) + v(x^2 + y^2).$$

The sixth also is an adjoint symmetry of type $\mathrm{d}^V F + J^*\mathrm{d}^H F$, but $L = \Gamma(F) = v^2$ is a 'degenerate Lagrangian'. Extending the search to time-dependent solutions, one obtains a further adjoint symmetry, which corresponds to the first integral $F_6 = \varphi - vt$. Needless to say, this is a very simple example: the differential equations coming from Γ can in fact be completely integrated; the first integrals $(F_1, F_2, F_3, F_4, F_6)$ determine the general solution.

Our second example is taken from [7], and is the fourth example in that paper: it concerns a non-holonomically constrained free particle. Here, the configuration manifold is $E = \mathbf{R} \times \mathbf{R}^3$ with coordinates (t, x, y, z) and the constraint manifold C is given by $\dot{z} = y\dot{x}$. Taking fibre coordinates $u = \dot{x} \circ \iota$, $v = \dot{y} \circ \iota$, the dynamical vector field on C is given by

$$\Gamma = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + yu\frac{\partial}{\partial z} - \frac{yuv}{1+y^2}\frac{\partial}{\partial u}.$$

We use Giachetta's choice of a projection here, which comes from the free particle Lagrangian in the sense of (5):

$$(P_i^a) = \begin{pmatrix} (1+y^2)^{-1} & 0 & y(1+y^2)^{-1} \\ 0 & 1 & 0 \end{pmatrix}.$$

The connection is

$$(\Gamma_i^a) = \begin{pmatrix} -yv(1+y^2)^{-2} & yu(1+y^2)^{-1} & v(1+y^2)^{-2} \\ 0 & 0 & 0 \end{pmatrix}.$$

A calculation similar to that carried out for the previous example yields the adjoint equations

$$\Gamma^2(\alpha_1) - 2yv(1+y^2)^{-1}\Gamma(\alpha_1) + 2y^2v^2(1+y^2)^{-2}\alpha_1 + v\beta = 0,$$
$$\Gamma^2(\alpha_2) - uv(1+y^2)^{-2}\alpha_1 - u\beta = 0,$$
$$\Gamma(\beta) + yv(1+y^2)^{-1}\beta - 2yv^2(1+y^2)^{-3}\alpha_1 = 0.$$

Looking for time-independent solutions where the α_a are functions of the base variables only and β is linear in u, v we again use Maple to find four solutions giving rise to independent first integrals:

(i) $\alpha_1 = 0, \quad \alpha_2 = 1, \quad \beta = 0;$
(ii) $\alpha_1 = -(1+y^2), \quad \alpha_2 = z, \quad \beta = 2v(1+y^2)^{-1};$
(iii) $\alpha_1 = -(1+y^2)^{1/2}, \quad \alpha_2 = 0, \quad \beta = v(1+y^2)^{-3/2};$

(iv) $\alpha_1 = -(1+y^2)^{1/2} \operatorname{arcsinh} y, \quad \alpha_2 = x,$
$\beta = -vy(1+y^2)^{-1} + (1+y^2)^{-3/2} \operatorname{arcsinh} y.$

The corresponding first integrals are:

$$F_1 = v, \qquad F_2 = -u(1+y^2) + vz, \qquad F_3 = -u(1+y^2)^{1/2},$$
$$F_4 = -u(1+y^2)^{1/2} \operatorname{arcsinh} y + vx.$$

Note that only two first integrals were given in [7], namely $\frac{1}{2}(F_1^2 + F_3^2)$ and $-F_3$. A fifth, time-dependent first integral, $F_5 = y - vt$, may be found easily by inspection from the expression for the dynamical vector field, and thus the equations can again be completely integrated. In the course of the analysis, we also find a few adjoint symmetries which satisfy the assumptions of Theorem 2, but do not produce a first integral; they lead to degenerate type functions L, and so are not of great interest.

An example of an adjoint symmetry giving rise to a non-degenerate Lagrangian can be found in [13], for the classical problem of a vertically rolling disc.

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